2.3: linear equations

Examples: $\boxed{\text{first order}}$ $\frac{dy}{dx} + y = 2$

$$x\frac{dy}{dx} + x^2 y = e^x$$

generic pattern: $a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$

$$\boxed{\text{standard form: } \frac{dy}{dx} + P(x)y = g(x)}$$

$P(x) = \frac{a_0(x)}{a_1(x)}$ $\quad g(x) = \frac{f(x)}{a_1(x)}$

we want to solve $\frac{dy}{dx} + P(x)y = g(x)$

note: the left hand side multiplied by the function $\mu(x)$ becomes $\mu(x)\frac{dy}{dx} + P(x)\mu(x)y = g(x)\mu(x)$

From the Product Rule

$$\Rightarrow \frac{d}{dx}\left[\mu(x)y\right] = \mu(x)\frac{dy}{dx} + \boxed{\frac{d\mu(x)}{dx}}y$$

$$\mu(x)\frac{dy}{dx} + \boxed{P(x)\mu(x)}y$$

$$P(x)\mu(x) = \frac{d\mu(x)}{dx}$$

$$\int P(x)dx = \int\frac{d\mu(x)}{\mu(x)} \Rightarrow \int P(x)dx = \ln|\mu(x)|$$

$$e^{\int P(x)dx} = e^{\ln|\mu(x)|} \Rightarrow \boxed{\mu(x) = e^{\int P(x)dx}}$$

Integration Factor

Ex) Page 61 #2 $\frac{dy}{dx} + 2y = 0$ $P(x) = 2$

$$\mu(x) = e^{\int P(x)dx}$$
$$\mu(x) = e^{\int 2dx}$$
$$\mu(x) = e^{2x}$$

$\boxed{\text{for Integration Factor } c = 0}$

$$\underbrace{e^{2x}\frac{dy}{dx} + 2y \cdot e^{2x}} = 0 \cdot e^{2x}$$

$$\frac{d}{dx}\left[\underset{\mu(x)}{\underset{\downarrow}{e^{2x}}} \cdot \underset{y}{\underset{\downarrow}{y}}\right] = 0$$

$$\int \frac{d}{dx}\left[e^{2x} y\right] = \int 0\,dx \Rightarrow e^{2x} y = C$$

$$\boxed{y = \frac{C}{e^{2x}} = Ce^{-2x}}$$

Ex) #25 $\frac{dy}{dx} = x + 5y$ $y(0) = 3$

$$\frac{dy}{dx} - \underset{P(x) = -5}{\underset{\uparrow}{5}}y = \underset{q(x) = x}{\underset{\uparrow}{x}}$$

Process: 1) rearrange in standard form
2) Identify $P(x)$ & $q(x)$
3) Find $\mu(x) = e^{\int P(x)dx}$
4) Multiply all terms by $\mu(x)$ to get the $\frac{d}{dx}[\mu(x) y] = q(x)\mu(x)$
5) Integrate
$\Rightarrow \mu(x) y = \frac{\int q(x)\mu(x)dx}{\mu(x)}$

$$\frac{dy}{dx} - 5y = x$$

$$\mu(x) = e^{\int -5\,dx} = e^{-5x}$$

$$\underbrace{e^{-5x}\frac{dy}{dx} - 5e^{-5x}y}_{\frac{d}{dx}\left[e^{-5x}y\right]} = x e^{-5x}$$

$$\frac{d}{dx}\left[e^{-5x}y\right] = x e^{-5x}$$

$$e^{-5x}y = \int x e^{-5x}\,dx$$

Parts:

	D	I
$+$	x	e^{-5x}
$-$	1	$-\frac{1}{5}e^{-5x}$
$+$	0	$\frac{1}{25}e^{-5x}$

$$-\frac{1}{5}x e^{-5x} - \frac{1}{25}e^{-5x} + C$$

$y(0) = 3$

$$\frac{e^{5x}y}{e^{-5x}} = \frac{-\frac{1}{5}x e^{-5x} - \frac{1}{25}e^{-5x} + C}{e^{-5x}}$$

$$y = -\frac{1}{5}x - \frac{1}{25} + Ce^{5x}$$

$$y(0) = -\frac{1}{5}\cancel{(0)} - \frac{1}{25} + Ce^{5(0)} = 3$$

$$-\frac{1}{25} + C = 3$$

$$C = \frac{75}{25} + \frac{1}{25}$$

$$\boxed{C = \frac{76}{25}}$$